On Solution of Homogeneous and Non-Homogeneous Differential Equation by Fractional Operators and N-Fractional Calculus Operator N^{υ}

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Abstract:- Fuchs type of homogeneous and non-homogeneous differential equation are solved by means of fractional operators and N-fractional calculus operators N^{υ}

1. INTRODUCTION

Despite detail of Nishimoto's fractional calculus, usually referred to as N-Fractional calculus, which can be found in Nishimoto (1984, 1987), we present an outline of notations employed in our investigation.

Let *f* = f(z) is an analytic function in the domain D(z ϵ D).

Then

Let *f* = *f*(*z*) is an analytic function in the domain *D*(*z* ϵ *D*). Then

$$f_{\upsilon} = (f)_{\upsilon} = {}_{c}(f)_{\upsilon}$$

$$= \frac{\Gamma(\upsilon+1)}{2\pi i}\int_{c}\frac{f(\zeta)}{(\zeta-z)^{\upsilon+1}}d\zeta \ , \quad \upsilon \in Z^{-}$$

and $(f)_{-m} = \lim_{\upsilon\to -m}(f)_{\upsilon}$,

such that where $-\pi \le \arg(\zeta - z) \le \pi$ for $\underset{-}{C}$, and $0 \le \arg(\xi - z) \le 2\pi$ for $\underset{+}{C}$, where $\zeta \ne z$ and moreover, $\underset{-}{C}$ and $\underset{+}{C}$ are curves along the cuts connecting *z* and $-\infty + i\text{Im}(z)$ and, *z* and $+\infty + i\text{Im}(z)$, respectively. $(f)_{\upsilon}$ is called fractional derivative of order $\upsilon\ (\upsilon > 0)$, where as it is called fractional integral of order $-\upsilon$ (υ<0), υ being arbitrary, of the function *f* with respect to *z*.

Let N^{V} is the *N*-fractional calculus operator, given by

$$N^{\upsilon} = \frac{\Gamma(\upsilon+1)}{2\pi i}\int_{c}\frac{f(\zeta)}{(\zeta-z)^{\upsilon+1}}d\zeta \ , \ \upsilon \notin Z^{-}$$

where

$$N^{-m} \lim_{\upsilon\to -m} N^{\upsilon} \ , \ m \notin Z^{+}$$

and the binary operator is defined by

$$N^{\beta} \circ N^{\alpha}(f) = N^{\beta} N^{\alpha}(f) = N^{\beta}(N^{\alpha} f), \quad \alpha, \beta \in R \quad .$$

Then the set $\{N^{\upsilon}\} = \{N^{\upsilon} \mid \upsilon \in R\}$ is an abelian product group, which possesses $(N^{\upsilon})^{-1} = N^{-\upsilon}$ as the inverse operator. In the present paper we have obtained the solution of homogeneous and non-homogeneous Fuchs type equations by using N^{υ} – fractional operator. The solution is best judged by the verifications given in the paper.

2. PARTICULAR SOLUTIONS OF HOMOGENEOUS EQUATIONS OF FUCHS TYPE

Theorem 1:

If $f \in F$ and $f_{-v} \neq 0$, then the non-homogeneous equation of Fuchs type

$$\phi_2 \cdot z + \phi_1 .(v - az) - \phi .av = f \tag{1}$$

has particular solution of the form

$$\phi = \left(\left(f_{-v} \cdot z^{-1} e^{-az}\right)_{-1} \cdot e^{az}\right)_{v-1} \tag{2}$$

where $\phi = \phi(z), f = f(z), z \in \mathbb{C}$, a = constant and v is arbitrary

$\mathbb{C}$ being a set of complex numbers, v is arbitrary and k being constant

Proof: we substitute

$$\phi = w_{v-1} \tag{3}$$

$$\phi_1 = w_v \tag{4}$$

$$\phi_2 = w_{v+1} \tag{5}$$

where $\phi_1 = \frac{d\phi}{dz}, \ \phi_2 = \frac{d^2\phi}{dz^2}, \ w = w(z)$ (6)

Now due to (3), (4) and (5), eq. (1) reduces to

$$w_{v+1}.z + w_v.(v - az) - w_{v-1}.av = f \tag{7}$$

which upon simplification, Nishimoto (1989), by employing

$$(p.q)_v = \sum_{n=0}^{\infty} \frac{\Gamma(v+1)}{\Gamma(v+1-n)\Gamma(n+1)} p_{v-n} q_n, \ p = p(z), q = q(z) \tag{8}$$

we obtain

$$(w_1 . z)_v + (w.(-az))_v = f$$

$$(w_1 . z - azw)_v = f \qquad (9)$$

$$w_1 . z - azw = f_{-v}$$

Now making a differintegration of the order –*v*, of (9) we obtain

$$w_1 - aw = \frac{f_{-v}}{z} \qquad (10)$$

Let the integrating factor of (10) be ρ ; and is expressed as

$$\rho = e^{-az}$$

and consequently, we obtain

$$w = \left(f_{-v} \cdot z^{-1} . e^{-az} \right)_{-1} \cdot e^{az} \qquad (11)$$

Eventually, substitution of (11), in (3), yields the particular solution as

$$\phi = w_{v-1} = \left(\left(f_{-v} \cdot z^{-1} . e^{-az} \right)_{-1} \cdot e^{az} \right)_{v-1} \qquad (12)$$

which is the required particular solution.

Justifications of the solution:

In case we employ (12) in L.H.S of (1), it readily reduces to

$$w_{v+1} . z + w_v . (v - az) - w_{v-1} . av$$

i.e. $(w_1 . z)_v + (w.(-az))_v$

which is

$$= (w_1 . z - azw)_v$$

obviously $(f_{-v})_v$ and that is *f*:

the significance of w_1 is as follows:

$$w_1 = \left(\left(f_{-v} . z^{-1} . e^{-az} \right)_{-1} . e^{-az} \right)_1$$

$$w_1 = f_{-v} \cdot z^{-1} + \left(f_{-v} \cdot z^{-1} . e^{-az} \right)_{-1} . \, a e^{az}$$

$$w_1 = \frac{f_{-v}}{z} + aw \qquad (13)$$

which completes and justifies the solution

Theorem 2:

'A' being the arbitrary constant of integration $\phi = \phi(z),\ z \in \mathbb{C}$

The particular solution of the homogeneous differential equations of the Fuchs type

$$\phi_2 \cdot z + \phi_1.(v - az) - \phi.av = 0 \tag{14}$$

has particular solution of the form

$$\phi = k\left(e^{az}\right)_{v-1} = k\,a^{v-1}e^{az} \tag{15}$$

Where $\phi = \phi(z), z \in \mathbb{C}$

a = constant and v is arbitrary

$\mathbb{C}$ being a set of complex numbers, v is arbitrary and k being constant.

Proof: we substitute

$$\phi = w_{v-1} \tag{16}$$

$$\phi_1 = w_v \tag{17}$$

$$\phi_2 = w_{v+1} \tag{18}$$

$$\phi_1 = \frac{d\phi}{dz},\ \phi_2 = \frac{d^2\phi}{dz^2},\ w = w(z) \tag{19}$$

Now due to (16), (17) and (18), eq. (14) reduces to

$$w_{v+1}.z + w_v.(v - az) - w_{v-1}.av = 0 \tag{20}$$

which upon simplification, Nishimoto (1989), by employing

$$(p.q)_v = \sum_{n=0}^{\infty} \frac{\Gamma(v+1)}{\Gamma(v+1-n)\Gamma(n+1)} p_{v-n} q_n,\ p = p(z), q = q(z) \tag{21}$$

we obtain

$$(w_1.z)_v + (w.(-az))_v = 0$$

$$\left(w_1.z - azw\right)_v = 0$$

$$w_1.z - azw = 0 \tag{22}$$

Now solving the above eq. by variable separable method, we obtained

$$w = k\,e^{az} \tag{23}$$

where *k* is an arbitrary constant of integration,

$\phi = w_{v-1}$

$= (ke^{az})_{v-1}$

$$= k(e^{az})_{v-1} \tag{24}$$

which is the required particular solution of eq. (14)

Justifications of the solution:

In case we employ (24) in L.H.S of (14), it readily reduces to

$$w_{v+1}.z + w_v.(v - az) - w_{v-1}.av$$

i.e. $(w_1 \,.\, z)_v + (w.(-az))_v$

which upon obvious simplification, yields

which is,

$$= \left(w_1.z - azw\right)_v$$

So now we may completely write

$$\phi_2 \cdot z + \phi_1.(v - az) - \phi.av = \left(w_1.z - azw\right)_v$$

where

$$w = Ae^{az}$$

$$w_1 = \left(Ae^{az}\right)_1$$

$$w_1 = a.Ae^{az} \tag{25}$$

which eventually this is simplified to $(0)_v = 0$, which completes and justifies the solution.

3. NOW SOLUTION OF HOMOGENEOUS AND NON- HOMOGENEOUS DIFFERENTIAL EQUATION BY N-FRACTIONAL CALCULUS OPERATOR N^{υ}

Theorem 3:

Let $\phi \in \varphi = \{\phi : 0 \neq |\phi_\upsilon| < \infty , \upsilon \in R\}$ and $f \in \varphi = \{f : 0 \neq |f_\upsilon| < \infty , \upsilon \in R\}$, then the non-homogeneous equation of Fuchs type

$$\phi_2 .z + \phi_1 .(v - az) - \phi .av = f \qquad (1)$$

has particular solution of the form

$$\phi = \left(\left(f_{-v} \cdot z^{-1} e^{-az}\right)_{-1} \cdot e^{az}\right)_{v-1} \qquad (2)$$

where

$$\phi = \phi(z), f = f(z), z \in \mathbb{C}$$

$\mathbb{C}$ being a set of complex numbers, v is arbitrary and k being constant.

Proof: We employ the fraction operator N^υ to both sides of (1),

$$N^\upsilon \{\phi_2 .z + \phi_1 .(v - az) - \phi .av\} = N^\upsilon \{f\}$$

$$\phi_{2+\upsilon} z + \phi_{1+\upsilon} .\upsilon + \phi_{1+\upsilon} .(v - \alpha z) + \phi .\upsilon(-a) - \phi_\upsilon .av = f_\upsilon$$

$$\phi_{2+\upsilon} z + \phi_{1+\upsilon} .(\upsilon + v - \alpha z) + \phi .\upsilon(-a\upsilon - av) = f_\upsilon \qquad (3)$$

Now, υ is taken such a way, that

$$-a\upsilon - av = 0$$

i.e. $\upsilon = -v$

substitute

$\upsilon = -v$ in (3), we get

$$\phi_{2-v} .z - \phi_{1-v} .az = f_{-v} \qquad (4)$$

Therefore, setting

$$\phi_{1-v} = w = w(z) \qquad (5)$$

$$\phi_{2-v} = w_1 \qquad (6)$$

we have

$$w_1 - w.a = f_{-v} .z^{-1} \qquad (7)$$

The particular solution of this first order linear differential equation is given by

$$\phi = \left(\left(f_{-v} \cdot z^{-1} e^{-az}\right)_{-1} \cdot e^{az}\right)_{v-1} \qquad (8)$$

Verification of Theorem 3:

Since $\phi = w_{v-1}$

$$\phi_1 = w_v$$

$$\phi_2 = w_{v+1}$$

$$w_{v+1}.z + w_v.(v - az) - w_{v-1}.av = f$$

which upon simplification, gives

$$w_1.z - azw = f_{-v}.z^{-1}$$

this being a linear differential equation of the first order, its solution is given by

$$w = \left(f_{-v} \cdot z^{-1}.e^{-az}\right)_{-1} \cdot e^{az}$$

hence we obtain

$$\phi = w_{v-1} = \left(\left(f_{-v} \cdot z^{-1}.e^{-az}\right)_{-1} \cdot e^{az}\right)_{v-1}$$

as a particular solution of (1).

Theorem 4:

Let $\phi \in \varphi = \{\phi : 0 \neq |\phi_\upsilon| < \infty, \ \upsilon \in R\}$ and $f \in \varphi = \{f : 0 \neq |f_\upsilon| < \infty, \upsilon \in R\}$, then the non-homogeneous equation of Fuchs type

$$\phi_2.z + \phi_1.(v - az) - \phi.av = 0 \tag{9}$$

has particular solution of the form

$$\phi = A(e^{az})_{v-1} \tag{10}$$

where

$$\phi = \phi(z), f = f(z), z \in \mathbb{C}$$

$\mathbb{C}$ being a set of complex numbers, *v* is arbitrary and *A* being constant.

Proof: We employ the fraction operator N^υ to both sides of (9),

$$N^\upsilon\{\phi_2.z + \phi_1.(v - az) - \phi.av\} = N^\upsilon\{0\}$$

$$\phi_{2+\upsilon}z + \phi_{1+\upsilon}.\upsilon + \phi_{1+\upsilon}.(v - \alpha z) + \phi.\upsilon(-a) - \phi_\upsilon.av = 0$$

$$\phi_{2+\upsilon} z + \phi_{1+\upsilon} .(\upsilon + v - \alpha z) + \phi .\upsilon(-a\upsilon - av) = 0 \quad (11)$$

Now, υ is taken such a way, that

$$-a\upsilon - av = 0$$

i.e. $\upsilon = -v$

substitute $\upsilon = -v$ in (11), we get

$$\phi_{2-\upsilon} z - \phi_{1-v} .az = 0 \quad (12)$$

Therefore, setting

$$\phi_{1-\upsilon} = w = w(z) \quad (13)$$

$$\phi_{2-\upsilon} = w_1 \quad (14)$$

we have

$$w_1 - w.a = 0 \quad (15)$$

The particular solution of this first order linear differential equation is given by

$$\phi = A.(e^{az})_{v-1} \quad (16)$$

where *A* is an arbitrary constant of integration.

Verification of Theorem 4:

Since $\phi = w_{v-1}$

$$\phi_1 = w_v$$

$$\phi_2 = w_{v+1}$$

$$w_{v+1}.z + w_v.(v - az) - w_{v-1}.av = 0$$

which upon simplification, gives

$$w_1.z - azw = 0$$

This being a linear differential equation of the first order, its solution is given by

$$w = A.e^{az}$$

hence we obtain

$$\phi = w_{v-1} = A.\left(e^{az}\right)_{v-1}$$

as a particular solution of (9).

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