Couette Flow Over A Deformable Permeable Bed

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Abstract-**The flow of a viscous fluid over a thin deformable porous layer fixed to the solid wall of a channel is considered. The upper solid wall moves with constant velocity U_0.The flow in the deformable porous layer is governed by a modified Darcy law based on binary mixture theory. The flow over the deformable porous layer is governed by Navier-Stokes equations. The expressions for the displacement of the porous medium and the fluid velocity are obtained on solving the governing equations in the free flow and porous flow regions. The effects of various physical parameters such as ¤^{f} and ¦ on the velocity and displacement are discussed in detail. When the thickness of the porous layer £ tends to zero and ¤$^{f} = 1$, the results obtained reduce to the classical ones of Yuan [1] for the Couette flow between parallel plates.**

Keywords: Couette flow; deformable porous layer; permeable bed

I. INTRODUCTION

Viscous flow through and past deformable porous media has been studied experimentally by many researchers with a view to understand some practical phenomena such as transpiration colling and gaseous diffusion in arteriar walls. Most of the research works available deal with flow through rigid porous media. But when a biofluid flows in a physiological system, such as blood vessel there will be an interaction between free flow and tissue regions. Thus the study of flow through and past a deformable porous layer is necessitated.

The study of deformation in porous materials with coupled fluid movement was initiated by Terzaghi [2] and later continued by Biot [3],[4],[5] and1956 into a successful theory of soil consolidation and acoustic propagation. Atkin and Craine (1976), Bowen (1980) and Bedford and Drumheller (1983) made important works on the theory of mixtures. Mow et al. (1984) developed a similar theory for the study of biological tissue mechanics. Using this theory arterial wall permeability is discussed by Jayaraman (1983). The same theory was also applied by Mow et al. (1985) and Holmes and Mow (1990) for the study of articular cartilages. Much of this analysis has been on one dimensional or purely radial compression without consideration of the influence of shear stresses on the deformable porous media.

Barry (1991) discussed the flow of a viscous fluid past a thin deformable porous layer. Rajasekhara, Rudraiah and Ramaiah(1975) discussed the Couette flow over a naturally permeable bed. Ranganatha and Siddagangamma (2004) studied a mathematical model for the blood flow in arteries assuming the artery as a symmetric channel with solid walls attached by a thin deformable porous layer. The aim of the present study is to revisit the problem solved by Rajasekhara et al. (1975) for Couette flow over a deformable porous layer. The problem is solved analytically and the results are deduced and discussed.

II. FORMULATION OF THE PROBLEM

The geometry consists of a steady, fully developed Couette flow through a channel with solid walls at $y = -L$ and $y = h$ and a porous layer of thickness L attached to the lower wall as shown in Fig.1. The flow region between the plates is divided into two layers. The flow region between the lower plate $y = -L$ and the interface $y = 0$ is termed as deformable porous region whereas the flow region between the interface $y = 0$ and the upper plate $y = h$ is designated as free flow region. The fluid velocity in the free flow region and porous flow region are assumed to be $(q,0,0)$ and $(v,0,0)$ respectively. The displacement due to the deformation of the solid matrix is taken as $(u,0,0)$. A pressure gradient $\frac{\partial p}{\partial x}(=G_0)$ is applied, producing an axially directed flow. Due to the assumption of an infinite channel, there is no x dependence in any of the terms except the pressure.

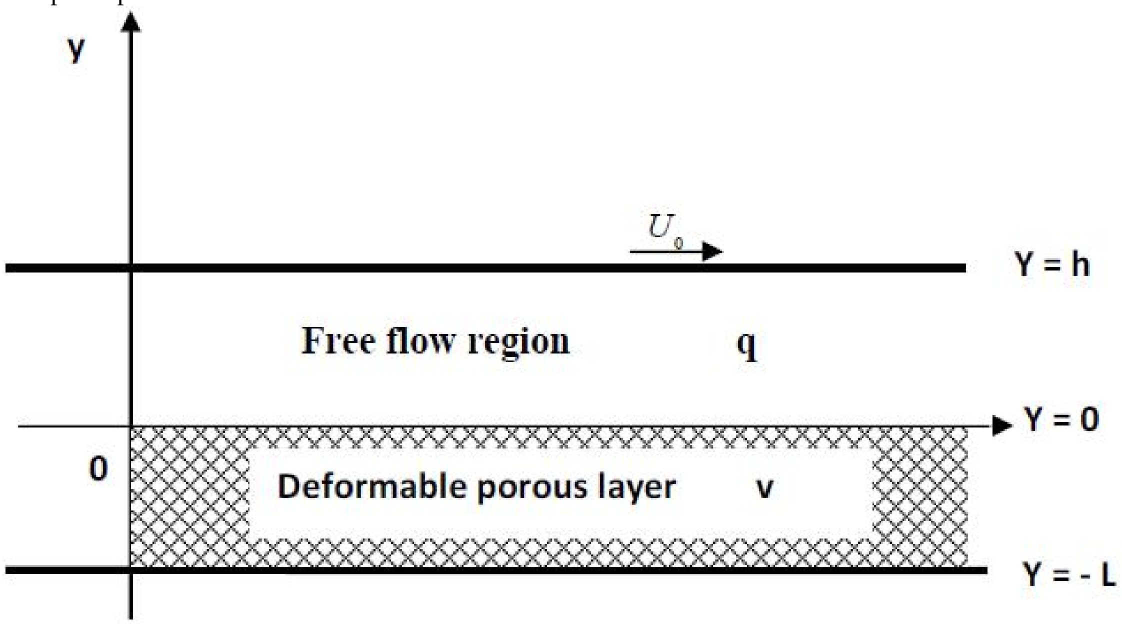


Fig.1 Physical Model.

With the assumptions mentioned above, the equations of motion in the free flow and deformable porous regions are [1991]

$$\text{«}\,\frac{\partial^2 u}{\partial y^2}=\text{¤}^{s}G_0-Kv \tag{1}$$

$$2\text{«}_a\,\frac{\partial^2 v}{\partial y^2}=\text{¤}^{f}G_0+K\,v \tag{2}$$

$$\text{«}_f\,\frac{\partial^2 q}{\partial y^2}=\frac{\partial p}{\partial x} \tag{3}$$

where «_a is the apparent viscosity of the fluid in the porous material, « is the Lame constant, «_f is the coefficient of viscosity, q is the flow velocity, u is the displacement, K is the drag coefficient, G_0 is the pressure gradient and ¤ is the volume fraction of component and $=s,f$ for the binary mixture of solid and fluid phases with $\text{¤}^s+\text{¤}^f=1$.

III. NON-DIMENSIONALIZATION OF THE FLOW QUANTITIES

It is convenient to introduce the following non-dimensional quantities.

$$u=\frac{-h^2G_0\hat{u}}{\text{«}},\ y=h\hat{y},\ v=\frac{-h^2G_0\hat{v}}{\text{«}_f},\ q=\frac{-h^2G_0\hat{q}}{\text{«}_f},\ U_0=\frac{-h^2G_0\hat{U}_0}{\text{«}_f},\ \text{¢}=\frac{Kh^2}{\text{«}_f},\ \text{£}=\frac{L}{h}$$

In view of the above dimensionless quantities, the equations (1) – (3) take the following form. The hats $(\wedge)$ are neglected here after.

$$\frac{d^2q}{dy^2}=-1 \tag{4}$$

$$\frac{d^2v}{dy^2}=\text{¢¦}\,v-\text{¤}^f\text{¦} \tag{5}$$

$$\frac{d^2u}{dy^2}=-\text{¢}\,v-\text{¤}^s \tag{6}$$

where $\text{¢}=\frac{Kh^2}{\text{«}_f},\text{¦}=\frac{\text{«}_f}{2\text{«}_a}$.

The parameter ¢ is a measure of the viscous drag of the outside fluid relative to drag in the porous medium. The parameter ¦ is the ratio of the bulk fluid viscosity to the apparent fluid viscosity in the porous layer.

The boundary conditions in non-dimensional form are

$$y=-\text{£}:\ v=0,u=0; \tag{7a}$$

$$y=0:\ \ q=\text{¤}^f v \tag{7b}$$

$$\frac{dq}{dy}=\frac{1}{\text{¦¤}^f}\frac{dv}{dy} \tag{7c}$$

$$\frac{dq}{dy}=\frac{1}{\text{¤}^s}\frac{du}{dy}\ . \tag{7d}$$

$$y=1:\ q=U_0 \tag{7e}$$

Equation (7b) equates the fluid velocity at the interface with the volume averaged velocity of the porous layer. The remaining two equations at the interface $y=0$ result from the conservation of axial momentum across the fluid-porous layer interface and the assumption that the proportion of the total stress in the porous layer borne by each component is proportional to its volume fraction.

IV. SOLUTION OF THE PROBLEM

Equations (4) to (6) are coupled differential equations that can be solved by using the boundary conditions (7). The displacement and velocities in free flow region and porous regions are obtained as

$$q(y) = -\frac{y^2}{2} + c_1 y + c_2 \tag{8}$$

$$v(y) = c_3 \cosh(\text{¸}\, y) + c_4 \sinh(\text{¸}\, y) + \frac{\text{··}^{\,f}}{\text{¢}} \tag{9}$$

$$u(y) = \frac{-c_3 \cosh(\text{¸}\, y)}{\text{¦}} - \frac{c_4 \sinh(\text{¸}\, y)}{\text{¦}} - \frac{y^2}{2} + c_5 y + c_6 \tag{10}$$

where $\text{¸}^{2} = \text{¢}\,\text{¦}$ and the constants c_1, c_2, c_3, c_4, c_5 and c_6 are given by

$$c_1 = a_2 c_4,\ c_2 = b_1 - c_1,\ c_3 = \frac{c_2 - b_2}{a_1},\ c_4 = \frac{-\left(a_1 b_3 + a_5 (b_2 - b_1)\right)}{a_1 a_6 + a_2 a_5},\ c_5 = \frac{c_1 + a_3 c_4}{a_4}\ ,$$

$$c_6 = c_3 a_7 - c_4 a_8 + a_9 c_5 + b_4,\ b_1 = U_0 + \frac{1}{2},\ a_1 = \text{¤}^{\,f},\ b_2 = \frac{\left(\text{¤}^{\,f}\right)^2}{\text{¢}}\ ,\ a_2 = \frac{\text{¸}}{\text{¦}\,\text{¤}^{\,f}},$$

$$a_3 = \frac{\text{¸}}{\text{¦}\,\text{¤}^{\,s}},\ a_4 = \frac{1}{\text{¤}^{\,s}}, a_5 = \cosh(\text{¸}\,\text{£}),\ a_6 = \sinh(\text{¸}\,\text{£}), b_3 = \frac{-\text{¤}^{\,f}}{\text{¢}},\ a_7 = \frac{a_5}{\text{¦}},$$

$$a_8 = \frac{a_6}{\text{¦}},\ a_9 = \text{£}\ \text{ and } b_4 = \frac{\text{£}^{2}}{2}. \tag{11}$$

V. RESULTS AND DISCUSSIONS

In this paper, the steady flow of a Couette flow over a thin deformable porous layer is investigated. When the thickness of the deformable porous layer $\text{£} \to 0$ and $\text{¤}^{\,f} \to 1$the expression for velocity given in (8) reduces to the result of Yuan[1976] for the Couette flow between two parallel rigid walls.

A. Mass flow rate

The dimensionless flow rate M per unit width of the channel in the free flow region is defined by

$$M = \int_0^1 q\, dy = \frac{-1}{6} + \frac{c_1}{2} + c_2. \tag{12}$$

Let F_p denote the fractional increase in mass flow rate through the plane Couette flow over that it would be if the flow were Poiseuillean.Then

$$F_p = \frac{M - M_p}{M_p}\ . \tag{13}$$

where M_p denotes the dimensionless mass flow rate of Poiseuille flow which is obtained from (13) by setting $U_0 = 0$. In other words

$$M_p = \frac{1}{3} + \frac{¸}{2¦¤^f}\left(\frac{\cosh(¸£)\left(\frac{\left(¤^f\right)^2}{¢} - \frac{1}{2}\right) - \frac{\left(¤^f\right)^2}{¢}}{¤^f \sinh(¸£) + \frac{¸}{¦¤^f}\cosh(¸£)}\right). \tag{14}$$

We note that F_p and M_p are functions of $¤^f$, $¢$, $¸$ and $£$.

The above analysis predicts the effect of deformable wall shear on the flow past a porous boundary. However to know the effect of the porous boundary it is necessary to compare the Couette flow with and without a porous boundary. This is done in the remaining part of this paper. If M_0 denotes the flux of Couette flow without a deformable layer and F_c denote the fractional increase in mass flow rate through the plane Couette flow with a permeable bed over what it would be if the flow were Couette flow without a permeable bed (i.e. $£ = 0, ¤^f = 1$) .Then

$$F_c = \frac{M - M_0}{M_0}, \text{ where } M_0 = \frac{U_0}{2} + \frac{1}{12} \tag{15}$$

B. *Maximum velocity*

The Maximum velocity in the free flow region is given by

$$q_m = \frac{-c_1^{\,2}}{2} + c_1^{\,2} + c_2 \;. \tag{16}$$

C. *Inference*

The solutions for the fluid velocities q, v in free flow region and deformable porous region and displacement of solid matrix u in the deformable porous region are evaluated numerically for different values of physical parameters such as the volume fraction of component $¤^f$, the viscous drag parameter $¢$, the viscosity parameter $¦$ and the thickness of lower wall $£$.

The variation of velocities and displacement in the channel q, v and u with y is calculated, from equations (8) to (10), for different values of U_0 and is shown in figures 2, 3 and 4 for fixed $¢$ =2.0, $¤^f$ =0.5, $¦$ =0.5 and $£$ =0.2. We observe that the velocities q, v and displacement u increase with the increases in the fluid velocity U_0.

The variation of velocities in the channel q, v with y is calculated, from equations (8) to (9), for different values of viscosity parameter $¦$ and is shown in figures 5 and 6 for fixed $¢$ =2.0, $¤^f$ =0.5, U_0 =1.0 and $£$ =0.2.We observe that the velocities q, v increases with the increase in viscosity parameter $¦$.

The variation of velocities in the channel q, v and u with y is calculated from equations (8) to (10), for different values of volume fraction of component $¤^f$ and is shown in figures 7, 8 and 9 for fixed $¢$ =2.0, $¦$ =0.5, U_0 =1.0 and $£$ =0.2.We observe that the velocities q, v increases with the increase in volume fraction of component $¤^f$ and the displacement u decreases with the increase in $¤^f$.

The variation of velocity q with y is calculated, from equation (8), for different values of $£$, $¤^f$ and is shown in figure 10 for fixed $¢$ =2.0, $¦$ =0.5 and U_0 =1.0. We observe that the velocity q increases with the increase in the thickness of the porous layer $£$. We also observe that the velocity in over the porous layer channel is more when compared with the velocity corresponding to the absence of the porous layer. Therefore the effect of porous layer is to enhance the velocity in the channel.

The variation of mass flow rate M with ¤^{f} is calculated, from equations (12), for different values of thickness of porous lining £ and is shown in figure 11 for fixed ¢ =2.0, ¦ =0.5 and U_0 =1.0. It is found that the mass flux M increases with the increase in the volume fraction coefficient ¤^{f}. We also observe that the mass flow rate increases with the increase in the thickness of porous lining £ .

The variation of mass flow rate M with ¤^{f} is calculated, from equations (12), for different values of upper plate velocity U_0 and is shown in figure 12 for fixed ¢ =2.0, ¦ =0.5 and £ =0.2.It is found that M increases with the increase in the volume fraction coefficient ¤^{f}. We also observe that the mass flow rate increases with the increase in the upper plate velocity U_0 .

The variation of fractional increase in mass flow rate F_p with ¤^{f} is calculated, from equations (14), for different values of upper plate velocity U_0 and is shown in figure 13 for fixed ¢ =2.0, ¦ =0.5 and £ =0.2. It is found that F_p increase with the increase in the volume fraction coefficient ¤^{f}.We also observe that the fractional increasing in Mass flow rate increases with the increase in the upper plate velocity U_0 .

The variation of fractional increase in mass flow rate F_p with ¤^{f} is calculated, from equations (14), for different values of thickness of porous lining £ and is shown in figure 14 for fixed ¢ =2.0, U_0 =1.0 and £ =0.2.It is found that F_p increase with the increase in the volume fraction coefficient ¤^{f}.We also observe that the fractional increasing increases in Mass flow rate with the decrease in the porous lining £ .

The variation of mass flow rate for Couette flow without deformable layer F_c with ¤^{f} is calculated, from equations (15), for different values of thickness of porous lining £ and is shown in figure 15 for fixed ¢ =2.0, U_0 =1.0 and ¦ =0.5. It is found that F_c increase with the increase in the volume fraction coefficient ¤^{f}.We also observe that the mass flow rate for Couette flow without deformable porous layer increases with the increases in the porous lining £ .

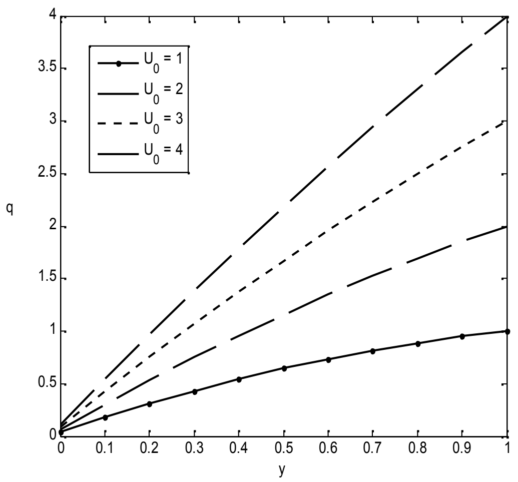


Fig 2. Velocity profiles of free flow region for different values of U_0 .

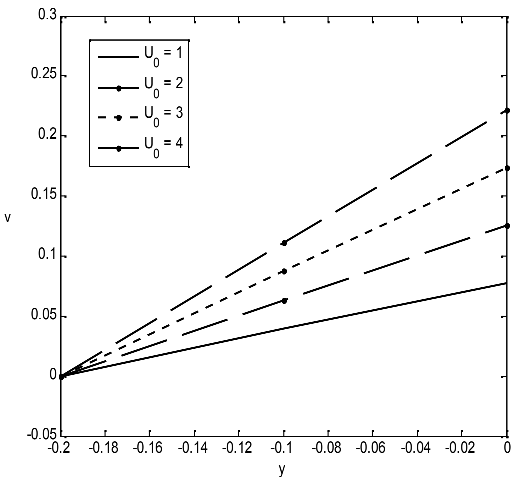


Fig 3.Velocity profiles of deformable porous layer for different values of U_0 .

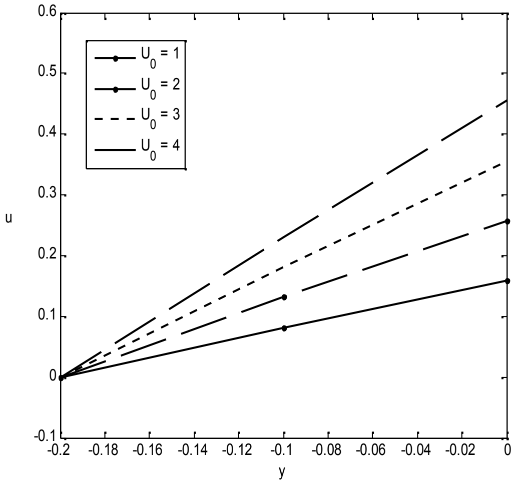


Fig 4.Displacement profiles in the deformable porous layer for different values of U_0.

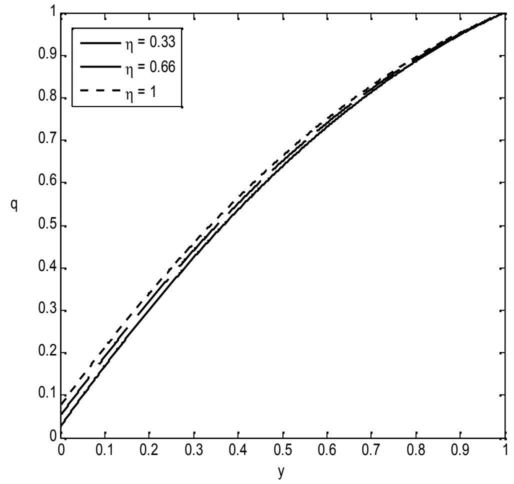


Fig 5.Velocity profiles in the free flow region for different values of η.

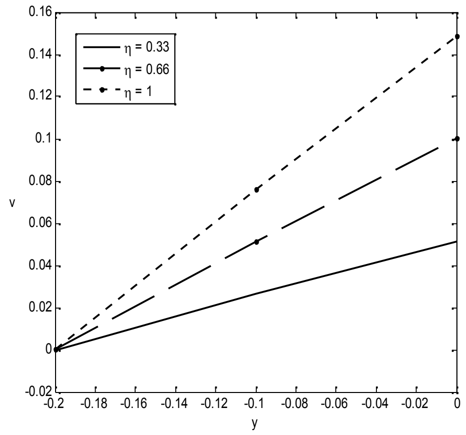
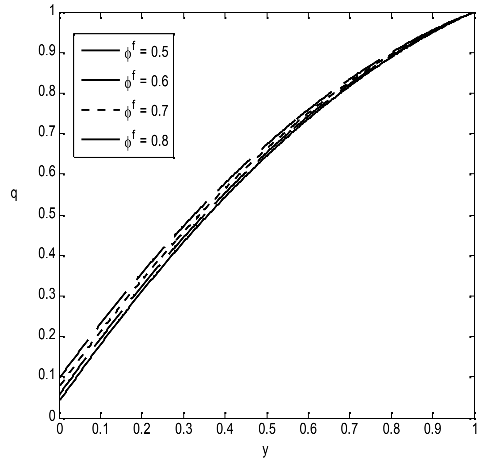


Fig 6.Velocity profiles in the deformable porous layer for different values of η.



g Fig7 .Velocity profiles in the free flow region for different values of ϕ^f.

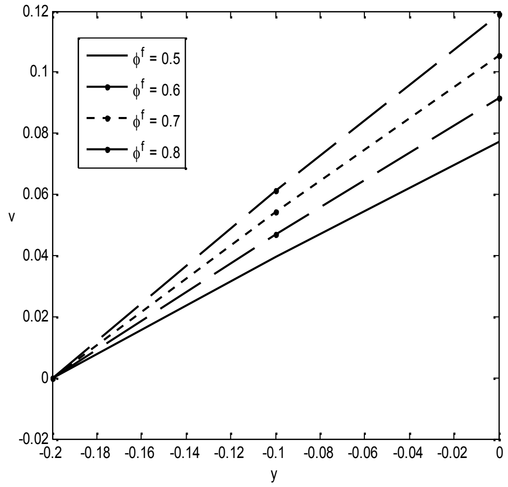


Fig 8.Velocity profiles in the deformable porous layer for different values of ϕ^f.

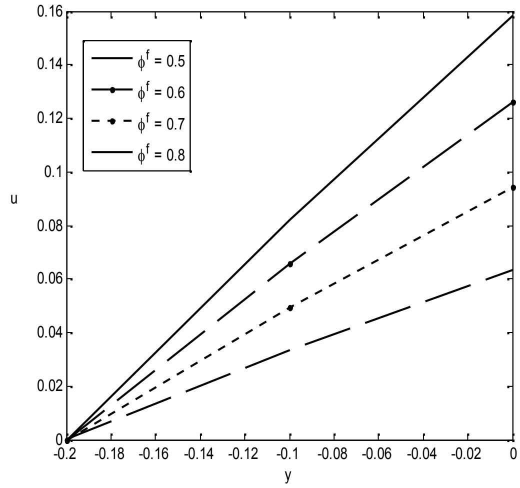


Fig 9.Displacement profiles in the deformable porous layer for different values of ϕ^f.

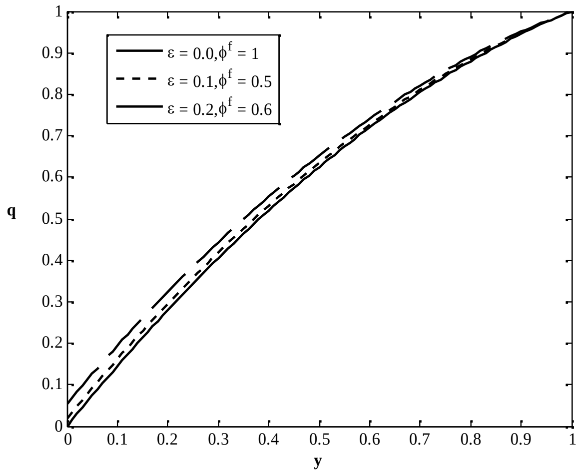


Fig 10. Velocity profiles in the free flow region for different values of £,¤^{f}

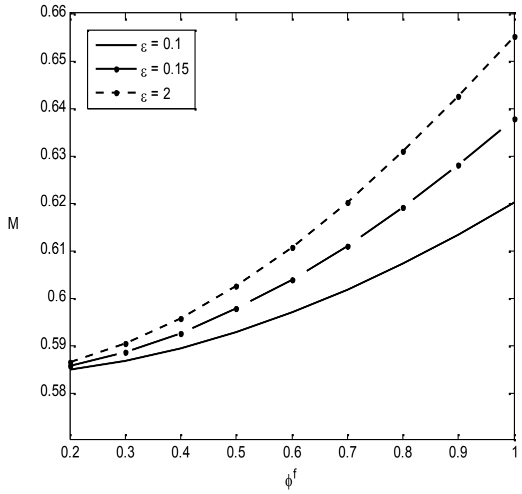


Fig 11.The variation of the Mass flow rate M in the free flow region for different values of £ .

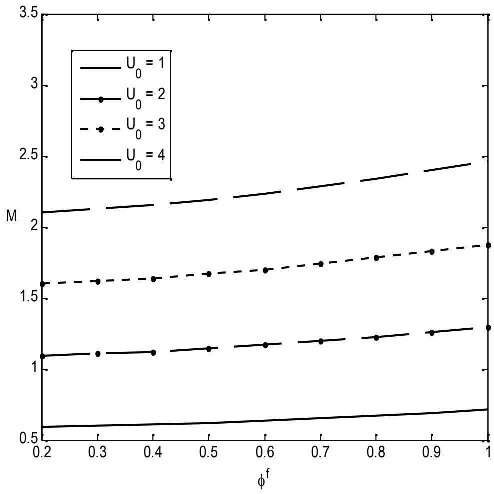


Fig 12. The variation of the Mass flow rate M in the free flow region for different values of U_0 .

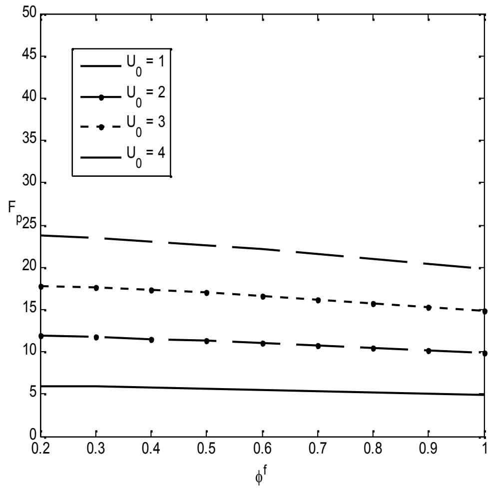


Fig 13. The variation of F_p with ¤^{f} for different values of U_0 .

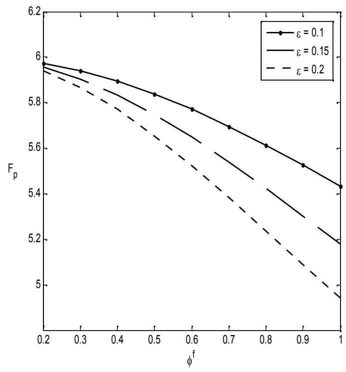


Fig 14. The variation of F_p with ¤^{f} for different values of £

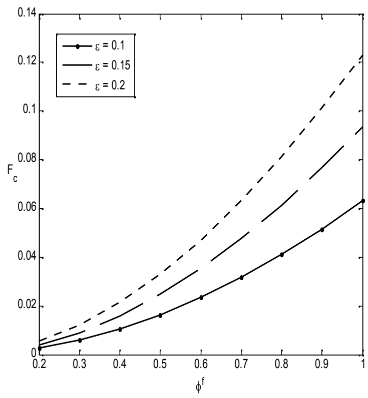


Fig15. The variation of F_c with ¤^{f} for different values of £

ACKNOWLEDGEMENT

The first two authors Prof. S.Sreenadh and M.Krishnamurthy expresses thanks to UGC for providing financial support through the Major Research Project to undertake this work.

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