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Estimation of Parameters in Cardioid Distribution From Censored Samples

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Abstract - **Rao Jammalamadaka and Vasudevan Mangalam (2009) introduced Censoring scheme in Circular data and in particular when the original data is assumed to be parametric model such as Von Mises, they discussed ML estimation of parameters and studied their large sample properties.**

Considering the above as the motivation, here, an attempt is made to tackle the estimation problem in the simplest [appears to be as there is complication of non-identifiability of Likelihood equation, $L(\rho,\mu)=L(-\rho,\mu+\pi)$**] model called Cardioid distribution as its pdf and cdf are in the closed forms on the lines of Girija (2010). She suggested a novel procedure of solving ML equations in case of Cardioid model for complete samples. The same methodology is adopted here in case of censored samples.**

Jeffrey (1961) introduced Cardioid distribution and used it to modeling directional spectra of ocean waves. Estimation of parameters from censored samples plays a very important role in statistical inference. Here an attempt is made to obtain M L estimators (rather M L equations as the estimators are not in the closed form) along with approximate variance covariance matrix of estimates of the parameters in the said distribution. A builtin function of MATLAB 7 called 'fmincon' is used to find ML estimates based on simulated data as suggested by Girija (2010). As the estimators are not in the closed form, simplified/ad hoc estimators are also proposed as an alternative to M L estimators, and studied their relative performance empirically.

Keywords - Estimators, censoring, nonidentifiable, simulation, fmincon

1. INTRODUCTION

This work deals with inferential aspects based on incomplete data. Particularly it deals with detailed discussion on ML estimation of parameters in Cardioid distribution which was proposed by Jeffrey (1961). This is a symmetric unimodal two-parameter distribution sometimes referred to as the Cosine distribution.

Probability density and distribution functions respectively are

$$f(\theta;\mu,\rho)=\frac{1}{2\pi}\left(1+2\rho\cos(\theta-\mu)\right) \tag{1}$$

$$F(\theta)=\frac{1}{2\pi}\left(\theta+2\rho\sin(\theta-\mu)+2\rho\sin\mu\right) \tag{2}$$

where $\theta,\mu\in[0,2\pi),\theta\geq\mu$ and $-\frac{1}{2}<\rho<\frac{1}{2}$

2. ML Estimation of Parameters From Censored Sample

Girija (2010) studied ML estimation of parameters from complete samples. Here an attempt is made to extend it to censored samples. In practical situations, one has to estimate parameters/parametric functions based on incomplete (censored) data.

In most of biological culture tests we come across left censored data while in the case of reliability studies, right censored data is commonly encountered. Therefore, it's always better to develop the most general case of doubly censored case from which the other two can be seen as particular cases.

Doubly Censored Sample

Let $\theta_{r+1} < \theta_{r+2} < .... < \theta_{n-s}$ be the available sample observations in a planned sample of size n after censoring r (fixed) and s (fixed) number of observations on left and right respectively.

The likelihood function $L(\theta;\mu,\rho)$ for doubly censored sample is

$$L \propto \left[F(\theta_{r+1})\right]^r \prod_{i=r+1}^{n-s} f(\theta_i)\left[1-F(\theta_{n-s})\right]^s \tag{3}$$

$$\log L \propto r\log\left[F(\theta_{r+1})\right] + \sum_{i=r+1}^{n-s} \log f(\theta_i) + s\log\left[1-F(\theta_{n-s})\right] \tag{4}$$

and the nonlinear M L equations to be solved iteratively are given by

$$\frac{\partial \log L}{\partial \mu} = g(\mu,\rho) \equiv \frac{2r\rho(\cos\mu - \cos(\theta_{r+1}-\mu))}{\theta_{r+1} + 2\rho(\sin(\theta_{r+1}-\mu)+\sin\mu)} + \sum_{i=r+1}^{n-s} \frac{2\rho\sin(\theta_i-\mu)}{1+2\rho\cos(\theta_i-\mu)} - \frac{2s\rho(\cos\mu - \cos(\theta_{n-s}-\mu))}{2\pi - \theta_{n-s} - 2\rho(\sin(\theta_{n-s}-\mu)+\sin\mu)} = 0 \tag{5}$$

$$\frac{\partial \log L}{\partial \rho} = h(\mu,\rho) \equiv \frac{2r(\sin\mu + \sin(\theta_{r+1}-\mu))}{\theta_{r+1} + 2\rho(\sin(\theta_{r+1}-\mu)+\sin\mu)} + \sum_{i=r+1}^{n-s} \frac{2\cos(\theta_i-\mu)}{1+2\rho\cos(\theta_i-\mu)} - \frac{2s(\sin\mu + \sin(\theta_{n-s}-\mu))}{2\pi - \theta_{n-s} - 2\rho(\sin(\theta_{n-s}-\mu)+\sin\mu)} = 0 \tag{6}$$

$$\begin{aligned}\frac{\partial}{\partial\mu} g(\mu,\rho) &= g_\mu(\mu,\rho) \\ &= \frac{2r\rho\left[-\theta_{r+1}\left(\sin\mu + \sin\left(\theta_{r+1}-\mu\right)\right) + 4\rho\left(\cos\theta_{r+1} - 1\right)\right]}{\left(\theta_{r+1} + 2\rho(\sin(\theta_{r+1}-\mu)+\sin\mu)\right)^2} \\ &+ \sum_{i=r+1}^{n-s} \frac{\left(-2\rho\cos(\theta_i-\mu) - 4\rho^2\right)}{\left(1+2\rho\cos(\theta_i-\mu)\right)^2} \\ &+ \frac{2s\rho\left[\left(2\pi-\theta_{n-s}\right)\left(\sin\mu + \sin\left(\theta_{n-s}-\mu\right)\right) + 4\rho\left(\cos\theta_{n-s} - 1\right)\right]}{\left(2\pi - \theta_{n-s} - 2\rho(\sin(\theta_{n-s}-\mu)+\sin\mu)\right)^2}\end{aligned} \tag{7}$$

$$\frac{\partial}{\partial\rho} g(\mu,\rho) = g_\rho(\mu,\rho)$$

$$= \frac{2r\theta_{r+1}\left(\cos\mu - \cos\left(\theta_{r+1}-\mu\right)\right)}{\left(\theta_{r+1} + 2\rho(\sin(\theta_{r+1}-\mu)+\sin\mu)\right)^2} + \sum_{i=r+1}^{n-s} \frac{2\sin(\theta_i-\mu)}{\left(1+2\rho\cos(\theta_i-\mu)\right)^2} - \frac{2s\left(2\pi-\theta_{n-s}\right)\left(\cos\mu - \cos\left(\theta_{n-s}-\mu\right)\right)}{\left(2\pi-\theta_{n-s} - 2\rho(\sin(\theta_{n-s}-\mu)+\sin\mu)\right)^2} \quad (8)$$

$$\frac{\partial}{\partial\rho} h(\mu,\rho) = h_\rho(\mu,\rho)$$

$$= \frac{-4r\left(\sin\mu + \sin\left(\theta_{r+1}-\mu\right)\right)^2}{\left(\theta_{r+1} + 2\rho(\sin(\theta_{r+1}-\mu)+\sin\mu)\right)^2} + \sum_{i=r+1}^{n-s} \frac{-4\cos^2(\theta_i-\mu)}{\left(1+2\rho\cos(\theta_i-\mu)\right)^2} - \frac{4s\left(\sin\mu + \sin\left(\theta_{n-s}-\mu\right)\right)^2}{\left(2\pi-\theta_{n-s} - 2\rho(\sin(\theta_{n-s}-\mu)+\sin\mu)\right)^2} \quad (9)$$

$$\frac{\partial}{\partial\mu} h(\mu,\rho) = h_\mu(\mu,\rho)$$

$$= \frac{2r\theta_{r+1}\left(\cos\mu - \cos\left(\theta_{r+1}-\mu\right)\right)}{\left(\theta_{r+1} + 2\rho(\sin(\theta_{r+1}-\mu)+\sin\mu)\right)^2} + \sum_{i=r+1}^{n-s} \frac{2\sin(\theta_i-\mu)}{\left(1+2\rho\cos(\theta_i-\mu)\right)^2} - \frac{2s\left(2\pi-\theta_{n-s}\right)\left(\cos\mu - \cos\left(\theta_{n-s}-\mu\right)\right)}{\left(2\pi-\theta_{n-s} - 2\rho(\sin(\theta_{n-s}-\mu)+\sin\mu)\right)^2} \quad (10)$$

Note

1. Substituting $r = s = 0$ in (5) and (6) reduces to correspond to complete sample case.

$$g(\mu,\rho) \equiv \sum_{i=1}^{n} \frac{2\rho\sin(\theta_i-\mu)}{1+2\rho\cos(\theta_i-\mu)} = 0 \quad (11)$$

$$h(\mu,\rho) \equiv \sum_{i=1}^{n} \frac{2\cos(\theta_i-\mu)}{1+2\rho\cos(\theta_i-\mu)} = 0 \quad (12)$$

2. Substituting $r = 0$ in (5) and (6) reduces to the results correspond to right censored can be obtained.

3. Substituting s = 0 in (5) and (6) reduces to the results correspond to left censored can be obtained.

In all the above cases, the variance covariance matrix is given by

$$(-J)^{-1} = \begin{pmatrix} -g_\mu & -g_\rho \\ -h_\mu & -h_\rho \end{pmatrix}^{-1}$$

3. Strategy Used For Maximizing LOG *L* of (4)

One way of optimizing log likelihood given in (4) is as follows

i) Obtaining stationary points by solving equations M.L. equations,
ii) Evaluating Hessian matrix at the stationary points and examining whether it is negative definite or not.

iii) If the Hessian matrix is negative definite then the stationary points in (ii) for a given sample will become ML estimates.

Though this method can be adopted, it converges to local maxima/global maxima depending on the choice of an initial approximation in the proximity of the final solution particularly when there exist several maxima/minima for the objective function under consideration. In this case, the objective function is nonidentifiable giving scope for the existence of multimodal case. It can be noted that in the Cardioid model $L(\rho,\mu)=L(-\rho,\mu+\pi)$. Therefore to identify the parameters uniquely we confine to **nonnegative values of** ρ **only** in simulation work to study various characteristics of the estimates.

For convenience, the problem of finding ML estimation is considered as an optimization problem whose mathematical formulation follows

$$\underset{(\mu,\rho)}{Max} \quad \log L(\mu,\rho \mid \; n,r,s,\theta_{r+1},\theta_{r+2},...,\theta_{n-s})$$

where $0 \le \mu < 2\pi$, $0 \le \rho < 0.5$ and

whose solution can easily be obtained by invoking built-in routines of MATLAB 7. μ_1 and ρ_1 denote the ML estimates obtained by suitably invoking MATLAB 7 function fmincon.

4. SIMULATION STUDY

Simulation work is conducted to

i) Obtain MLEs and to study the relative performance of MLEs in comparison with two more alternative estimators proposed in this section,

ii) Observe the relative performance with respect to the Angular Variance and MSE in estimating the mean direction while in estimating the concentration parameter a measure of dispersion of variance type and corresponding MSE are used and

iii) Present the final outcome in the form of a table for various combinations of n, r, s. Varying n=10, 20 and 30, for all possible combination of r and s such that (r + s) = n*10%, n*20% and n*30% censoring is applied to conduct Simulation Study.

Various Estimators Proposed for Simulation Study

The simulation study is conducted for $\mu=\frac{\pi}{4}$, and $\rho=0.1$ varying sample size *n* = 10, 20, 30; with 1000 simulation cycles in each case. The choice of the sample sizes and the true values μ_t and ρ_t of μ and ρ are made arbitrarily.

Computational Schema of Various Estimators

Notations used

- Let μ_t and ρ_t be true values of Cardioid parameters μ and ρ respectively. Here they are chosen arbitrarily as $\mu=\frac{\pi}{4}$, and $\rho=0.1$ and are used for data generation also.
- *n, r* and *s* respectively denote the sample size, number of observations censored on left and right respectively.
- Number of simulation cycle = 1000 in each case (for a combination of n, r and s).
- Let $X_o=\begin{pmatrix}\mu_o\\ \rho_o\end{pmatrix}=\begin{pmatrix}0.65\\ 0.08\end{pmatrix}$ as obtained from graph

- Let μ_i and ρ_i for $i = 1(1)1000$ be the solutions obtained based on data set $(\theta_{r+1} < \theta_{r+2} < ... < \theta_{n-r-s})$ by invoking the built-in function 'fmincon' of MATLAB 7 whose description was already given in the earlier sections.
- Let μ^* and ρ^* be the M.L. estimates and are computed as

$$\mu^* = \frac{1}{1000}\sum \mu_i \quad \text{and} \quad \rho^* = \frac{1}{1000}\sum \rho_i$$

- Computation alternative estimates called quadrant specific estimates denoted by $\hat{\mu}$ and $\hat{\rho}$ are computed as under

Let $C_i = \sum_{k=r+1}^{n-s} \cos\theta_k$ and $S_i = \sum_{k=r+1}^{n-s} \sin\theta_k$

$$\mu_i = \tan^{-1}\left(\frac{S_i}{C_i}\right) = \begin{cases} \tan^{-1}\left(\frac{S_i}{C_i}\right) & \text{if } C_i > 0\ ,\ S_i \geq 0 \\ \frac{\pi}{2} & \text{if } C_i = 0\ ,\ S_i > 0 \\ \tan^{-1}\left(\frac{S_i}{C_i}\right) + \pi & \text{if } C_i < 0 \\ \tan^{-1}\left(\frac{S_i}{C_i}\right) + 2\pi & \text{if } C_i \geq 0, S_i < 0 \\ \text{undefined} & \text{if } C_i = 0, S_i = 0 \end{cases}$$

$$\rho_i = \sqrt{C_i^2 + S_i^2} \quad \text{for } i = 1(1)1000$$

$$\hat{\mu} = \frac{1}{1000}\sum \mu_i \quad \text{and} \quad \hat{\rho} = \frac{1}{1000}\sum \rho_i$$

- Recall μ_i and ρ_i for $i = 1(1)1000$ obtained from 'fmincon' and using them, define

$$C_1 = \sum cos\mu_i \quad \text{and} \quad S_1 = \sum sin\mu_i$$

another alternative estimators denoted by $\tilde{\mu}$ and $\tilde{\rho}$ are as follows

$$\tilde{\mu} = \tan^{-1}\left(\frac{S_1}{C_1}\right) = \begin{cases} \tan^{-1}\left(\dfrac{S_1}{C_1}\right) & \text{if } C_1 > 0\ ,\ S_1 \geq 0 \\ \dfrac{\pi}{2} & \text{if } C_1 = 0\ ,\ S_1 > 0 \\ \tan^{-1}\left(\dfrac{S_1}{C_1}\right) + \pi & \text{if } C_1 < 0 \\ \tan^{-1}\left(\dfrac{S_1}{C_1}\right) + 2\pi & \text{if } C_1 \geq 0, S_1 < 0 \\ \text{undefined} & \text{if } C_1 = 0, S_1 = 0 \end{cases}$$

$$\tilde{\rho} = \sqrt{C_1^2 + S_1^2}$$

Now, we define various sampling characteristics w.r.t. μ^* and ρ^* as under

- Bias (μ^*) $= 1 - \cos\left(\mu^* - \mu_t\right)$.
- Unbiasing constant of μ^* is represented by ub (μ^*) = μ_t / μ^* .
- Angular variance of μ^* is denoted as AV(μ^*) and is given by

  AV(μ^*) = $\frac{2}{1000}\sum_{k=1}^{1000}\left[1 - \cos\left(\mu_1(k) - \mu^*\right)\right]$.
- Mean Square error of μ^* is denoted as MSE(μ^*) and is given by

  MSE(μ^*) = [Bias (μ^*)]^2 + AV(μ^*).
- Corresponding representations for $\hat{\mu}$ and $\tilde{\mu}$ are similar and computed in the same way.

The following measures are computed for the study of relative performance of ρ^* , $\hat{\rho}$ and $\tilde{\rho}$.

- Bias (ρ^*) = $\left|\rho_t - \rho^*\right|$.
- Unbiasing constant of ρ^* is represented by ub(ρ^*) = ρ_t / ρ^* .
- Variance type of measure of dispersion is denoted as

  V(ρ^*) = $\frac{1}{1000}\sum_{k=1}^{1000}\left[\rho_1(k) - \rho^*\right]^2$.
- Mean Square error of ρ^* is denoted as MSE(ρ^*) and is given by

  MSE (ρ^*) = [Bias (ρ^*)]^2 + V(ρ^*).
- Corresponding representations for $\hat{\rho}$ and $\tilde{\rho}$ are similar and computed in the same way.

The estimates and the measures for relative performance are presented in table 1.

The Ascending Order of Empirical MSEs of the Estimates for $\mu = \pi/4$ and $\rho = 0.1$ are presented in Table 2.

TABLE 1

Sampling Characteristics of $\mu = \pi/4$ and $\rho = 0.1$ for n = 10, r = 0,1,2 and s = 0,1,2 and corresponding results for n = 20, 30 are also computed but not presented here to save space.

n r s	μ	Bias (μ)	ub (μ)	AV (μ)	MSE (μ)	ρ	Bias (ρ)	ub (ρ)	V (ρ)	MSE (ρ)
10 0 0										
	2.7504	*1.3841*	*0.2856*	*2.2335*	*4.1491*	*0.2490*	*0.1490*	*0.4016*	*0.0208*	*0.0430*
	0.7817	*6.8761e-006*	*1.0047*	*0.4111*	*0.4111*	*0.2272*	*0.1272*	*0.4401*	*0.0336*	*0.0498*
	0.7376	*0.0011*	*1.0648*	*0.4095*	*0.4095*	*0.2267*	*0.1267*	*0.4411*	*0.0336*	*0.0497*
10 1 0										
	2.0643	*0.7122*	*0.3805*	*0.7885*	*1.2957*	*0.2565*	*0.1565*	*0.3899*	*0.0211*	*0.0456*
	0.8520	*0.0022*	*0.9218*	*0.5189*	*0.5189*	*0.2391*	*0.1391*	*0.4182*	*0.0360*	*0.0553*
	0.7856	*1.6068e-008*	*0.9998*	*0.5156*	*0.5156*	*0.1813*	*0.0813*	*0.5517*	*0.0393*	*0.0459*
10 0 1										
	2.6082	*1.2494*	*0.3011*	*1.6679*	*3.2288*	*0.2490*	*0.1490*	*0.4016*	*0.0215*	*0.0437*
	0.7962	*5.8054e-005*	*0.9865*	*0.4286*	*0.4286*	*0.2266*	*0.1266*	*0.4413*	*0.0331*	*0.0492*
	0.7490	*6.6133e-004*	*1.0486*	*0.4268*	*0.4268*	*0.0763*	*0.0237*	*1.3103*	*0.0557*	*0.0563*
10 0 2										
	2.3801	*1.0239*	*0.3300*	*1.1714*	*2.2197*	*0.2415*	*0.1415*	*0.4140*	*0.0202*	*0.0402*
	0.8136	*3.9804e-004*	*0.9653*	*0.4694*	*0.4694*	*0.2313*	*0.1313*	*0.4323*	*0.0339*	*0.0511*
	0.7569	*4.0693e-004*	*1.0377*	*0.4670*	*0.4670*	*0.0091*	*0.0909*	*10.965*	*0.0832*	*0.0915*
10 2 0										
	3.3044	*1.8124*	*0.2377*	*1.6057*	*4.8904*	*0.2434*	*0.1434*	*0.4109*	*0.0203*	*0.0409*
	0.8065	*2.2236e-004*	*0.9739*	*0.4350*	*0.4350*	*0.2301*	*0.1301*	*0.4346*	*0.0339*	*0.0508*
	0.7533	*5.1469e-004*	*1.0426*	*0.4327*	*0.4327*	*0.1279*	*0.0279*	*0.7821*	*0.0443*	*0.0451*
10 1 1										
	2.8411	*1.4661*	*0.2764*	*1.3917*	*3.5411*	*0.2542*	*0.1542*	*0.3934*	*0.0212*	*0.0450*
	0.8036	*1.6497e-004*	*0.9774*	*0.4372*	*0.4372*	*0.2277*	*0.1277*	*0.4391*	*0.0332*	*0.0496*
	0.7545	*4.7733e-004*	*1.0410*	*0.4353*	*0.4353*	*0.3363*	*0.2363*	*0.2974*	*0.0450*	*0.1009*

TABLE 2

Ascending Order of Empirical MSEs of the Estimates for $\mu = \pi/4$ and $\rho = 0.1$ and Preference of the estimator for *n* =10 corresponding results for *n* =20 and *n* = 30 are also computed but not presented here.

S No.	n r s	Mean Direction μ	Concentration ρ	Preference of the estimator	
				Mean Direction μ	Concentration ρ
1	*10 0 0*	*MSE($\tilde{\mu}$) MSE(μ^*) MSE($\hat{\mu}$)*	*MSE($\hat{\rho}$) MSE($\tilde{\rho}$) MSE(ρ^*)*	$\tilde{\mu}$	$\hat{\rho}$
2	*10 1 0 10 %*	*MSE($\tilde{\mu}$) MSE(μ^*) MSE($\hat{\mu}$)*	*MSE($\hat{\rho}$) MSE($\tilde{\rho}$) MSE(ρ^*)*	$\tilde{\mu}$	$\hat{\rho}$
3	*10 0 1 10%*	*MSE($\tilde{\mu}$) MSE(μ^*) MSE($\hat{\mu}$)*	*MSE($\hat{\rho}$) MSE(ρ^*) MSE($\tilde{\rho}$)*	$\tilde{\mu}$	$\hat{\rho}$
4	*10 0 2 20%*	*MSE($\tilde{\mu}$) MSE(μ^*) MSE($\hat{\mu}$)*	*MSE($\hat{\rho}$) MSE(ρ^*) MSE($\tilde{\rho}$)*	$\tilde{\mu}$	$\hat{\rho}$
5	*10 2 0 20%*	*MSE($\tilde{\mu}$) MSE(μ^*) MSE($\hat{\mu}$)*	*MSE($\hat{\rho}$) MSE($\tilde{\rho}$) MSE(ρ^*)*	$\tilde{\mu}$	$\hat{\rho}$
6	*10 1 1 20%*	*MSE($\tilde{\mu}$) MSE(μ^*) MSE($\hat{\mu}$)*	*MSE($\hat{\rho}$) MSE(ρ^*) MSE($\tilde{\rho}$)*	$\tilde{\mu}$	$\hat{\rho}$
7	*10 3 0 30%*	*MSE($\tilde{\mu}$) MSE(μ^*) MSE($\hat{\mu}$)*	*MSE($\hat{\rho}$) MSE(ρ^*) MSE($\tilde{\rho}$)*	$\tilde{\mu}$	$\hat{\rho}$
8	*10 0 3 30%*	*MSE($\tilde{\mu}$) MSE(μ^*) MSE($\hat{\mu}$)*	*MSE($\hat{\rho}$) MSE(ρ^*) MSE($\tilde{\rho}$)*	$\tilde{\mu}$	$\hat{\rho}$
9	*10 2 1 30%*	*MSE($\tilde{\mu}$) MSE(μ^*) MSE($\hat{\mu}$)*	*MSE($\hat{\rho}$) MSE(ρ^*) MSE($\tilde{\rho}$)*	$\tilde{\mu}$	$\hat{\rho}$
10	*10 1 2 30%*	*MSE($\tilde{\mu}$) MSE(μ^*) MSE($\hat{\mu}$)*	*MSE($\hat{\rho}$) MSE(ρ^*) MSE($\tilde{\rho}$)*	$\tilde{\mu}$	$\hat{\rho}$

REFERENCES

[1] Dattatreya Rao, A.V., Ramabhadra Sarma, I. and Girija, S.V.S. (2007)." On Wrapped Version of Some Life Testing Models", Comm Statist., - Theor.Meth. 36, 11, pp.2027-2035.

[2] Dattatreya Rao, A.V., Girija, S.V.S. and V.J. Devaraaj, (2009). "Tests for Circular Uniformity", Journal contemplated by ANU, 151-164.

[3] Dattatreya Rao A.V., Girija S.V.S. and Phani, Y. (2011a). "Differential Approach to Cardioid Distribution", Computer Engineering and Intelligen Systems, ISSN 2222-1719, Vol 2, No.8, pp.1-6.

[4] Dattatreya Rao A.V., Girija S.V.S. and Radhika A.J.V. (2011b), "A Note on Offset Cauchy Distribution", Proceedings of the 5th International Conference of IMBIC on Mathematical Sciences for Advancement of Science and Technology. Kolkata, pp. 133- 139.

[5] Dattatreya Rao A.V., Girija S.V.S. and Phani, Y. (2011c). "On Stereographic Logistic Model", Proceedings of NCAMES, AU Engineering College, Visakhapatnam, pp. 139 – 141.

[6] Dattatreya Rao A.V., Girija S.V.S. and V.J. Devaraaj (2013a), "On The Rising Sun Wrapped Lognormal and The Rising Sun Wrapped Exponential Models", International Journal of Statistics and Systems, Volume 3, Number 1, pp. 1-10 .

[7] Dattatreya Rao A.V., Girija S.V.S. and V.J. Devaraaj (2013b), "On Characteristics of Wrapped Gamma Distribution", (2013), IRACST – Engineering Science and Technology: An International Journal (ESTIJ), ISSN: 2250-3498, Vol.3, No.2, p. 228 – 232.

[8] Dattatreya Rao A.V., Girija S.V.S. and Phani, Y. (2013c), "Arc Tan- Exponential Type Distribution Induced By Stereographic Projection / Bilinear Transformation on Modified Wrapped Exponential Distribution", Journal of the Applied Mathematics, Statistics and Informatics (JAMSI), Vol.9,No.1.

[9] Devaraaj V. J. (2012), Some Contributions to Circular Statistics, Thesis submitted to Acharya Nagarjuna University for the award of Ph. D.

[10] Fisher, N. I. (1993). Statistical Analysis of Circular Data. Cambridge University Press, Cambridge.

[11] Girija, S.V.S. (2010).Construction of New Circular Models, VDM VERLAG, Germany. ISBN 978-3-639-27939-9.

[12] Girija S.V.S., Phani, Y. and Dattatreya Rao A.V., (Sep. 2013a). "On Stereographic Lognormal Distribution", International Journal of Advances in Applied Sciences (IJAAS), Vol 2 No. 3.

[13] Girija S.V.S., Radhika A J V and Dattatreya Rao A.V., (May 2013b)," On Bimodal Offset Cauchy Distribution", Journal of the Applied Mathematics, Statistics and Informatics (JAMSI), Vol.9, No. 1.

[14] Girija S.V.S., Radhika A J V and Dattatreya Rao A.V., (2014), On Offset l-Arc Models, Mathematics and Statistics 2(3): 127-136.

[15] Jammalamadaka, S.R. and Sen Gupta, A. (2001). Topics in Circular Statistics, World Scientific Press, Singapore.

[16] Jeffreys, H., (1961). Theory of Probability, 3rd edition. Oxford University Press.

[17] Mardia, K.V. and Jupp, P.E. (2000). Directional Statistics, John Wiley, Chic ester.

[18] Phani, Y., Girija S.V.S and Dattatreya Rao A.V., (2012). "Circular Model Induced by Inverse Stereographic Projection On Extreme-Value Distribution," IRACST – Engineering Science and Technology: An International Journal (ESTIJ), ISSN: 2250-3498,Vol.2, No. 5, pp. 881 – 888.

[19] Phani, Y., Girija S.V.S. and Dattatreya Rao A.V., (2013a). "On Construction of Stereographic Semicircular Models", Journal of Applied Probability and Statistics, Vol. 8, no. 1, pp. 75-90.

[20] Phani, Y. (2013b). On Stereographic Circular and Semicircular Models, Thesis submitted to Acharya Nagarjuna University for the award of Ph.D.

[21] Radhika A J V, Girija S.V.S. and Dattatreya Rao A.V., (2013), "On Univariate Offset Pearson Type II Model – Application To Live Data", International Journal of Mathematics and Statistics Studies, Vol.1, No. 1, 2013, pp.1-9,

[22] Radhika A J V, (2014), "Mathematical Tools in the Construction of New Circular Models", unpublished thesis submitted to Acharya Nagarjuna University for the award of Ph.D.

[23] Ramabhadra Sarma, I., Dattatreya Rao, A.V. and Girija, S.V.S., (2009). "On Characteristic Functions of Wrapped Half Logistic and Binormal Distributions", International Journal of Statistics and Systems, 4(1), 33–45.

[24] Ramabhadra Sarma, I., Dattatreya Rao, A.V. and Girija, S.V.S., (2011)." On Characteristic Functions of Wrapped Lognormal and Weibull Distributions", Journal of Statistical Computation and Simulation Vol. 81(5), 579–589.

[25] Rao, J.S. and Mangalam, V. (2009). A general censoring scheme for circular data, Statistical Methodology ,6 , 280-289