On Solution of Homogeneous and Non-Homogeneous Differential Equation by Fractional Operators and N-Fractional Calculus Operator $N^\nu$

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Abstract: Fuchs type of homogeneous and non-homogeneous differential equation are solved by means of fractional operators and N-fractional calculus operators $N^\nu$

1. INTRODUCTION

Despite detail of Nishi’smoto’s fractional calculus, usually referred to as N-Fractional calculus, which can be found in Nishimoto (1984, 1987), we present an outline of notations employed in our investigation.

Let $f = f(z)$ is an analytic function in the domain $D(z \in D)$.

Then

Let $f = f(z)$ is an analytic function in the domain $D(z \in D)$. Then

$$f_\nu = (f)_\nu = c(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}} d\zeta, \; \nu \in \mathbb{Z}^-$$

and $(f)_m = \lim_{\nu \to m} (f)_\nu$,

such that where $-\pi \leq \arg(\zeta - z) \leq \pi$ for $C$, and $0 \leq \arg(\xi - z) \leq 2\pi$ for $C$, where $\zeta \neq z$ and moreover, $C$ and $C_\pm$ are curves along the cuts connecting $z$ and $-\infty + i\text{Im}(z)$ and, $z$ and $+\infty + i\text{Im}(z)$, respectively. $(f)_\nu$ is called fractional derivative of order $\nu$ $(\nu > 0)$, where as it is called fractional integral of order $-\nu$ $(\nu < 0)$, $\nu$ being arbitrary, of the function $f$ with respect to $z$.

Let $N^\nu$ is the N-fractional calculus operator, given by

$$N^\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}} d\zeta, \; \nu \in \mathbb{Z}^-$$

where

$$N^{-m} \lim_{\nu \to m} N^\nu, \; m \notin \mathbb{Z}^+$$
and the binary operator is defined by

\[ N^\beta \circ N^\alpha (f) = N^\beta (N^\alpha f), \quad \alpha, \beta \in R \]

Then the set \( \{ N^\nu \} = \{ N^\nu \mid \nu \in R \} \) is an abelian product group, which possesses \((N^\nu)^{-1} = N^{-\nu}\) as the inverse operator. In the present paper we have obtained the solution of homogeneous and non-homogeneous Fuchs type equations by using \( N^\nu \) – fractional operator. The solution is best judged by the verifications given in the paper.

2. PARTICULAR SOLUTIONS OF HOMOGENEOUS EQUATIONS OF FUCHS TYPE

**Theorem 1:**

If \( f \in F \) and \( f_{-\nu} \neq 0 \), then the non-homogeneous equation of Fuchs type

\[ \phi_2 \cdot z + \phi_1 (v - az) - \phi. av = f \]

has particular solution of the form

\[ \phi = \left( (f_{-\nu} \cdot z^{-\nu} e^{-\nu})_{-1} \cdot e^{\nu z} \right)_{-1} \]

where \( \phi = \phi(z), f = f(z), z \in \mathbb{C}, a = \text{constant and } \nu \) is arbitrary

\( \mathbb{C} \) being a set of complex numbers, \( v \) is arbitrary and \( k \) being constant.

**Proof:** we substitute

\[ \phi = w_{v-1} \]

\[ \phi_1 = w_v \]

\[ \phi_2 = w_{v+1} \]

where

\[ \phi_1 = \frac{d\phi}{dz}, \phi_2 = \frac{d^2\phi}{dz^2}, w = w(z) \]

Now due to (3), (4) and (5), eq. (1) reduces to

\[ w_{v+1} \cdot z + w_v \cdot (v - az) - w_{v-1} \cdot av = f \]

which upon simplification, Nishimoto (1989), by employing

\[ (p \cdot q)_v = \sum_{n=0}^{\infty} \frac{\Gamma(v+1)}{\Gamma(v+1-n)\Gamma(n+1)} p_{v-n} q_n, \quad p = p(z), q = q(z) \]

we obtain
\[(w_1 \cdot z)_v + (w(-az))_v = f\]

\[(w_1 \cdot z - azw)_v = f\]  \hspace{1cm} (9)

\[w_1 \cdot z - azw = f_{-v}\]

Now making a differintegration of the order \(-v\), of (9) we obtain

\[w_1 - aw = \frac{f_{-v}}{z}\]  \hspace{1cm} (10)

Let the integrating factor of (10) be \(\rho\); and is expressed as

\[\rho = e^{-aw}\]

and consequently, we obtain

\[w = \left(f_{-v} \cdot z^{-1}e^{-aw}\right)_{-1} \cdot e^{aw}\]  \hspace{1cm} (11)

Eventually, substitution of (11), in (3), yields the particular solution as

\[\phi = w_{v-1} = \left(f_{-v} \cdot z^{-1}e^{-aw}\right)_{v-1}\]  \hspace{1cm} (12)

which is the required particular solution.

**Justifications of the solution:**

In case we employ (12) in L.H.S of (1), it readily reduces to

\[w_{v+1} - w_v (v - az) - w_{v-1} \cdot av\]

i.e. \[(w_1 \cdot z)_v + (w(-az))_v\]

which is

\[= (w_1 \cdot z - azw)\]

obviously \((f_{-v})_v\) and that is \(f\):

the significance of \(w_1\) is as follows:

\[w_1 = \left(f_{-v} \cdot z^{-1}e^{-aw}\right)_{-1}\]

\[w_1 = f_{-v} \cdot z^{-1} + \left(f_{-v} \cdot z^{-1}e^{-aw}\right)_{-1} \cdot a e^{aw}\]

\[w_1 = \frac{f_{-v}}{z} + aw\]  \hspace{1cm} (13)
which completes and justifies the solution

**Theorem 2:**

‘A’ being the arbitrary constant of integration \( \phi = \phi(z), \quad z \in \mathbb{C} \)

The particular solution of the homogeneous differential equations of the Fuchs type

\[
\phi_2 \cdot z + \phi_1.(v - az) - \phi.a v = 0 \quad (14)
\]

has particular solution of the form

\[
\phi = k\left(e^{az}\right)_{v-1} = k a^{-v} e^{av} \quad (15)
\]

Where \( \phi = \phi(z), \quad z \in \mathbb{C} \)

\( a = \) constant and \( v \) is arbitrary

\( \mathbb{C} \) being a set of complex numbers, \( v \) is arbitrary and \( k \) being constant.

**Proof:** we substitute

\[
\phi = w_{v-1} \quad (16)
\]

\[
\phi_1 = w_v \quad (17)
\]

\[
\phi_2 = w_{v+1} \quad (18)
\]

\[
\phi_1 = \frac{d\phi}{dz}, \quad \phi_2 = \frac{d^2\phi}{dz^2}, \quad w = w(z) \quad (19)
\]

Now due to (16), (17) and (18), eq. (14) reduces to

\[
w'_{v+1} . z + w_v . (v - az) - w_{v-1} . a v = 0 \quad (20)
\]

which upon simplification, Nishimoto (1989), by employing

\[
(p.q)_v = \sum_{n=0}^{\infty} \frac{\Gamma(v+1)}{\Gamma(v+1-n)\Gamma(n+1)} p_{v-n} q_n, \quad p = p(z), q = q(z) \quad (21)
\]

we obtain

\[
(w_1 . z)_v + (w_1 . (az))_v = 0
\]

\[
(w_1 . z - az)_v = 0
\]

\[
w_1 . z - az w = 0
\]
Now solving the above eq. by variable separable method, we obtained

\[ w = ke^{aw} \quad (23) \]

where \( k \) is an arbitrary constant of integration,

\[ \phi = w_{v-1} \]

\[ = (ke^{aw})_{v-1} \]

\[ = k(e^{aw})_{v-1} \quad (24) \]

which is the required particular solution of eq. (14)

**Justifications of the solution:**

In case we employ (24) in L.H.S of (14), it readily reduces to

\[ w_{v+1}z + w_v(v - az) - w_{v-1}av \]

i.e. \( (w_1 . z) + (w_1 (-az))_v \)

which upon obvious simplification, yields

which is,

\[ = (w_1 . z - azw)_v \]

So now we may completely write

\[ \phi_2 . z + \phi_1 (v - az) - \phi_1 av = (w_1 . z - azw)_v \]

where

\[ w = Ae^{aw} \]

\[ w_1 = (Ae^{aw})_v \]

\[ w_1 = aAe^{aw} \quad (25) \]

which eventually this is simplified to \( (0)_v = 0 \), which completes and justifies the solution.

**3. NOW SOLUTION OF HOMOGENEOUS AND NON-HOMOGENEOUS DIFFERENTIAL EQUATION BY N-FRACTIONAL CALCULUS OPERATOR** \( N^\nu \)
Theorem 3:

Let $\phi \in \varphi = \left\{ \phi : 0 \neq |\phi| < \infty, \nu \in R \right\}$ and $f \in \varphi = \left\{ f : 0 \neq |f| < \infty, \nu \in R \right\}$, then the non-homogeneous equation of Fuchs type

$$\phi_2z + \phi_1(v - az) - \phi av = f$$

has particular solution of the form

$$\phi = \left( \left( f_{-v} \cdot z^{-1} e^{-ac} \right)_{-1} \cdot e^{ac} \right)_{-1}$$

where

$$\phi = \phi(z), f = f(z), z \in C$$

being a set of complex numbers, $v$ is arbitrary and $k$ being constant.

Proof: We employ the fraction operator $N^\nu$ to both sides of (1),

$$N^\nu \left\{ \phi_2z + \phi_1(v - az) - \phi av \right\} = N^\nu \left\{ f \right\}$$

$$\phi_2 + \phi_1, v + \phi_1, v, (v - az) + \phi_\nu(-a) - \phi_\nu av = f_\nu$$

Now, $\nu$ is taken such a way, that

$$-av - av = 0$$

i.e. $\nu = -v$

substitute

$$\nu = -v$$

in (3), we get

$$\phi_{2-v}, z - \phi_{1-v}, az = f_{-v}$$

Therefore, setting

$$\phi_{1-v} = w = w(z)$$

$$\phi_{2-v} = w_1$$

we have

$$w_1 - w \cdot a = f_{-v} z^{-1}$$

The particular solution of this first order linear differential equation is given by

$$\phi = \left( \left( f_{-v} \cdot z^{-1} e^{-ac} \right)_{-1} \cdot e^{ac} \right)_{-1}$$
Verification of Theorem 3:

Since \( \phi = w_{v-1} \)

\[ \phi_1 = w_v \]

\[ \phi_2 = w_{v+1} \]

\[ w_{v+1} \cdot z + w_v \cdot (v - az) - w_{v-1} \cdot av = f \]

which upon simplification, gives

\[ w_1 \cdot z - azw = f_{-v} \cdot z^{-1} \]

this being a linear differential equation of the first order, its solution is given by

\[ w = \left( f_{-v} \cdot z^{-1} e^{-av} \right)_v \cdot e^{av} \]

hence we obtain

\[ \phi = w_{v-1} = \left( \left( f_{-v} \cdot z^{-1} e^{-av} \right)_{v-1} \cdot e^{av} \right) \]

as a particular solution of (1).

Theorem 4:

Let \( \phi \in \varphi = \{ \phi : 0 \neq |\phi| < \infty, \nu \in \mathbb{R} \} \) and \( f \in \varphi = \{ f : 0 \neq |f_{\nu}| < \infty, \nu \in \mathbb{R} \} \), then the non-homogeneous equation of Fuchs type

\[ \phi_{v+1} \cdot z + \phi_1 \cdot (v - az) - \phi_2 \cdot av = 0 \]

has particular solution of the form

\[ \phi = A (e^{av})_{v-1} \]

where

\[ \phi = \phi(z), f = f(z), z \in \mathbb{C} \]

\( \mathbb{C} \) being a set of complex numbers, \( v \) is arbitrary and \( A \) being constant.

Proof: We employ the fraction operator \( N^v \) to both sides of (9),

\[ N^v \{ \phi_2 \cdot z + \phi_1 \cdot (v - az) - \phi_2 \cdot av \} = N^v \{ 0 \} \]

\[ \phi_{2+1} \cdot z + \phi_{1+u} \cdot v + \phi_{1+u} \cdot (v - az) + \phi_2 \cdot av - \phi_2 \cdot av = 0 \]
\[
\phi_{z_1} z + \phi_{v_1} (v + v - \alpha z) + \phi_{v} (-a v - a v) = 0
\]  \hspace{1cm} (11)

Now, \( U \) is taken such a way, that

\[-a v - a v = 0\]

i.e. \( \nu = -v \)

substitute \( U = -v \) in (11), we get

\[
\phi_{z_1} z - \phi_{v_1} a z = 0
\]  \hspace{1cm} (12)

Therefore, setting

\[
\phi_{v_1} = w = w(z)
\]  \hspace{1cm} (13)

\[
\phi_{z_1} = w_1
\]  \hspace{1cm} (14)

we have

\[
w_1 - w a = 0
\]  \hspace{1cm} (15)

The particular solution of this first order linear differential equation is given by

\[
\phi = A \left( e^{az} \right)_{v_{-1}}
\]  \hspace{1cm} (16)

where \( A \) is an arbitrary constant of integration.

**Verification of Theorem 4:**

Since \( \phi = w_{v_{-1}} \)

\[
\phi_1 = w_v
\]

\[
\phi_2 = w_{v+1}
\]

\[
w_{v+1} z + w_z (v - a z) - w_{v_{-1}} a v = 0
\]

which upon simplification, gives

\[
w_1 z - a z w = 0
\]

This being a linear differential equation of the first order, its solution is given by

\[
w = Ae^{av}
\]

hence we obtain
\[ \phi = w_{v-1} = A \left( e^{az} \right)_{v-1} \]

as a particular solution of (9).

References

6. **Chena Ram and Pushpa Choudhary:** On solutions of homogeneous and non-homogeneous Legendre and Euler equations by fractional operators \( N^\nu \).